INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS
Int. J. Numer. Meth. Fluids 2004; **44**:1377–1388 (DOI: 10.1002/fld.604)

Optimal control of material concentration using Fourier series

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SUMMARY

This paper presents an optimal control of the material concentration using Fourier series and finite element method. It is assumed that the optimal control value can be expanded into a Fourier series. The Fourier coefficient is identified to minimize the performance function and the optimal control value is determined. The Sakawa–Shindo algorithm is used for the minimization algorithm. The advection–diffusion equation and shallow water equation are used for the analysis of material concentration and water flow. The Crank–Nicolson scheme and finite element method using bubble function element with stabilized control parameter are employed as temporal and special discretization. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: parameter identification; Fourier series; finite element method

1. INTRODUCTION

On December 8th in 1998, the Japanese Ministry of the Environment announced that the achievement rate of environmental standards of lakes and marches of COD (chemical oxygen demand) was 41.0% in Japan. Moreover, the rate in 1997 decreased 1% compared with that in 1996. To solve such a serious social problem, the Japanese Ministry of Land, Infrastructure and Transport has a plan of the water conduction [1] to clear up water quality. The aim of the water conduction plan is that the river water is conducted into the enclosed water area to promote water exchange and to clear up the water quality in the final stage. To do this, how to estimate the volume of inflow for the water conduction is important. To evaluate the water conduction plan, the optimal control theory is suitable with the finite element computation [2].

In this research, the shallow water equation and the advection–diffusion equation are used for the material concentration and water flows. For the discretization, the finite element method based on the bubble function element with stabilized control parameter [3] is used. To determine the value of the water conduction, the optimal control theory is employed. However, to calculate the optimal control value, a large amount of the computational storage is required. To reduce the computational storage, the main idea of the present paper is that the optimal

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Received 19 October 2002
Revised 4 May 2003

control value is expanded into a Fourier series and a parameter identification technique is used to determine the Fourier coefficients. After the verification based on the simple numerical examples, the present method is applied to the water conduction plan of the lake Kasumigaura in Japan. It is shown that the present method is useful and stable method for the computation of the water conduction problem.

2. BASIC EQUATION

The shallow water equation is used to calculate the water flow in the enclosed water area. Using the summation convention and denoting the differentiation with respective to co-ordinate x_i $(i=1,2)$ by $(\ )_{,i}$, the equations of motion and continuity can be expressed as follows:

$$\dot{u}_i + u_j u_{i,j} + g\eta_{,i} - \nu(u_{i,j} + u_{j,i})_{,j} + f u_i = 0 \quad (1)$$

$$\dot{\eta} + \{(h+\eta)u_i\}_{,i} = 0 \quad (2)$$

where u_i is the flow rate, g is the gravitational acceleration, η is the water elevation, h is the water depth as shown in Figure 1, ν is the coefficient of kinematic eddy viscosity, which is represented as: $\nu=(\kappa_c/6)u_*(h+\eta)$, f is the coefficient of bottom friction, which is represented as: $f=u_*/(h+\eta)$, u_* is the friction velocity, which is given as $u_*=gn^2/(h+\eta)^{1/3}(u_k u_k)^{1/2}$, where n is the Manning roughness coefficient and κ_c is the Kalman coefficient.

To calculate the flow of material concentration, the advection–diffusion equation is used. Equation of conservation can be expressed using the summation convention:

$$\dot{c} + u_i c_{,i} - \kappa c_{,ii} = 0 \quad (3)$$

where c is the material concentration and κ is the diffusion coefficient.

The Crank–Nicolson scheme and the finite element method using bubble function element are employed for the temporal and spatial discretization.

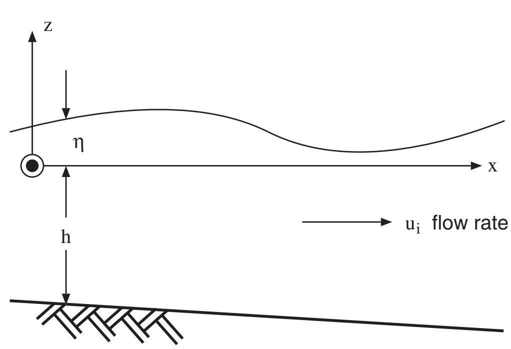

Figure 1. Co-ordinate.

3. CONTROL THEORY

The optimal control theory is applied to solve the unsteady optimal control problem. However, this method requires large computational storage for calculation. Because, to calculate optimal control value, it is necessary to store the state value at all time stage and at all spatial nodal points, the load for computer is very high and it is very difficult to calculate an actual problem or a large scale problem. To solve such a problem, it is assumed that the optimal control value is expressed by the Fourier series. Parameter identification technique is applied to determine the Fourier coefficient and the optimal control value.

3.1. Performance function

In the inverse problem, the performance function should be introduced. It is the quadratic sum of the difference between the computed and required state values. The formulation of parameter identification theory resolves itself into a minimization problem of the performance function. The performance function is represented as follows:

$$J(\bar{\mathbf{U}}) = \frac{1}{2}\int_T (U - U_{\mathrm{obj}})^{\mathrm{T}} Q (U - U_{\mathrm{obj}})\,\mathrm{d}t \tag{4}$$

$$\mathbf{U} = \{u_i, \eta, c_i\}^{\mathrm{T}} \tag{5}$$

where $\mathbf{U}$ and $\mathbf{U}_{\mathrm{opt}}$ mean the computed and required state values, respectively, in which $\bar{\mathbf{U}}$ represents the control value at the control point, T is total time and Q is the weighting matrix.

3.2. Fourier series

In this research, optimal control value is expressed by the Fourier series as shown in Equation (6). However, it is very difficult to determine many Fourier coefficients which are expressed by Equations (7), (8). Therefore, all Fourier coefficients are treated as unknown parameters. Parameter identification technique is applied to determine the Fourier coefficients of the optimal control value $\bar{u}_i(t)$.

$$\bar{u}_i(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2\pi nt}{T}\right) + b_n \sin\left(\frac{2\pi nt}{T}\right)\right) \tag{6}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \bar{u}_i(t)\cos\left(\frac{2\pi nt}{T}\right)\mathrm{d}t \tag{7}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \bar{u}_i(t)\sin\left(\frac{2\pi nt}{T}\right)\mathrm{d}t \tag{8}$$

4. PARAMETER IDENTIFICATION

To determine the Fourier coefficients, a parameter identification technique is used as the control problem. It has an advantage of reducing the computational storage requirement because calculation proceeds only as forward time marching way. Therefore, it is not necessary to store all state values. The parameter identification problem is regarded as a minimization problem of the performance function J. To apply the Sakawa–Shindo method as a minimization algorithm, the performance function is modified as follows:

$$K^l = J^l + \tfrac{1}{2}(\mathbf{a}_k^{(l+1)} - \mathbf{a}_k^{(l)})^{\mathrm{T}}\mathbf{c}^{(l)}\,\tfrac{1}{2}(\mathbf{a}_k^{(l+1)} - \mathbf{a}_k^{(l)}) \tag{9}$$

where (l) is the iteration cycle of the Sakawa–Shindo method, $\mathbf{c}^{(l)}$ is the weighting matrix, and $\mathbf{a}_k$ is the Fourier coefficient which is treated as an unknown value as follows:

$$\mathbf{a}_k = \{a_0, a_1, \ldots, a_n, b_1, b_2, \ldots, b_n\}^{\mathrm{T}} \tag{10}$$

where $\mathbf{c}^{(l)}$ is renewed every iteration of the Sakawa–Shindo method. The modified performance function is differentiated by $\mathbf{a}_k$ and thus the following equation is derived:

$$\left[\frac{\partial K^l}{\partial \mathbf{a}_k}\right] = \int_T \left[\frac{\partial \mathbf{U}}{\partial \mathbf{a}_k}\right]^{\mathrm{T}} Q(\mathbf{U} - \mathbf{U}_{\mathrm{obj}})\,\mathrm{d}t + \mathbf{c}^{(l)}(\mathbf{a}_k^{(l+1)} - \mathbf{a}_k^{(l)}) \tag{11}$$

From the stationary condition $\partial K^l/\partial \mathbf{a}_k = 0$, the identified Fourier coefficients are renewed by the following equation:

$$\mathbf{a}_k^{(l+1)} = \mathbf{a}_k^{(l)} - \mathbf{c}^{-1(l)}\left[\int_{t_0}^{t_f}\left\{\left[\frac{\partial \mathbf{U}}{\partial \mathbf{a}_k}\right]^{\mathrm{T}} Q(\mathbf{U} - \mathbf{U}_{\mathrm{obj}})\right\}\mathrm{d}t\right] \tag{12}$$

4.1. Sensitivity equation

The most difficult problem faced in this research is how to calculate $[\partial \mathbf{U}/\partial \mathbf{a}_k]$, which is called as a sensitivity matrix. To calculate the sensitivity matrix, all basic equations are differentiated as follows:

$$\frac{\partial}{\partial \mathbf{a}_k}\{\dot{u}_i + u_j u_{i,j} + g\eta_{,i} - v(u_{i,j} + u_{j,i})_{,j} + f u_i\} = 0 \tag{13}$$

$$\frac{\partial}{\partial \mathbf{a}_k}\{\dot{\eta} + \{(h+\eta)u_i\}_{,i}\} = 0 \tag{14}$$

$$\frac{\partial}{\partial \mathbf{a}_k}\{\dot{c} + u_i c_{,i} - \kappa c_{,ii}\} = 0 \tag{15}$$

All differentiated equations are discretized and calculated with the same scheme as the basic equations.

5. MINIMIZATION ALGORITHM

5.1. Sakawa–Shindo method

In this research, the Sakawa–Shindo method [4] is used for the minimization algorithm. To apply the Sakawa–Shindo method, the modified performance function is derived as in Equation (12) using stabilized constants $\mathbf{c}^{(l)}$.

The calculation algorithm of the Sakawa–Shindo method is summarized as follows:

1. Set $l = 0$, and assume the initial identified vector $\mathbf{a}_k^{(0)}$.
2. Solve the initial state vector $u_i^{(0)}$, $\eta^{(0)}$, $c^{(0)}$ using Equations (1)–(3).
3. Solve the initial performance function $J^{(0)}$ using Equation (4).
4. Solve the sensitivity matrix $[\partial K^l/\partial \mathbf{a}_k]$ using Equations (13)–(15).
5. Solve the identified vector $\mathbf{a}_k^{(l+1)}$ using Equation (12).
6. Compute the error norm $e = \|\mathbf{a}_j^{(l+1)} - \mathbf{a}_j^{(l)}\|$, and if $e \leqslant \varepsilon$ then stop, else go to 7.
7. Solve the initial state vector $u_i^{(l+1)}$, $\eta^{(l+1)}$, $c^{(l+1)}$ using Equations (1)–(3).
8. Solve the initial performance function $J^{(l+1)}$ using Equation (4).
9. The weighting parameter $\mathbf{c}^{(l)}$ is changed as follows:
   If $J^{(l+1)} \leqslant J^{(l)}$ then $\mathbf{c}^{(l+1)} = 0.9\mathbf{c}^{(l)}$, $l+1 \rightarrow l$ an go to 4, else $\mathbf{c}^{(l)} = 2.0\mathbf{c}^{(l)}$, and go to 5.

6. NUMERICAL EXAMPLE

6.1. Rectangular model

To verify the present method, two numerical examples are carried out. Target parameters are determined and the water conduction value and target material concentration at the objective point are calculated using the determined parameters. The target concentration is evaluated at the objective point. Plate 1 shows the domain used in the computation.

6.1.1. Case 1. In case 1, parameter identification of one control variable is tested. The optimal control value is expressed as in Equation (16) and parameter a_0 is determined as $a_0 = -0.748656$. The target concentration at the objective point is calculated using parameter a_0. Then, the method of this research is tested using the target concentration.

In Figure 2, it is shown that the result of this calculation achieves the target concentration. The identified parameter is $a_0 = -0.733848$, the error is 1.98%. From these results, this method is shown as the useful tool for the computation.

$$\bar{u}_i(t) = a_0 \tag{16}$$

6.1.2. Case 2. In case 2, parameter identification of 10 variables is carried out. The optimal control value is expressed by Equation (17) and each parameter $\mathbf{a}_k$ should be determined. Target concentration at the objective point is calculated using parameter $\mathbf{a}_k$. In the next step, the present method is tested using the target concentration.

In Figure 3, the convergence of the performance function is shown. In Figures 4–7, the results of this calculation are represented, those achieve the target concentration but the identified parameter has some errors. The minimum error of the identified parameter is 7.16%

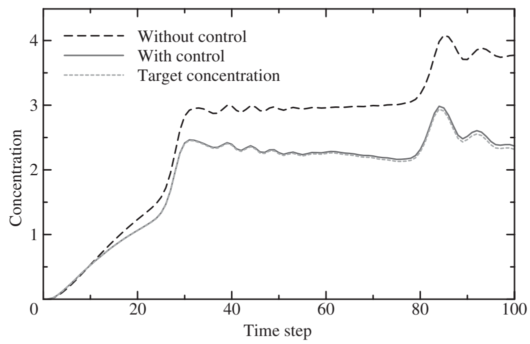

Figure 2. Variation of concentration at objective point.

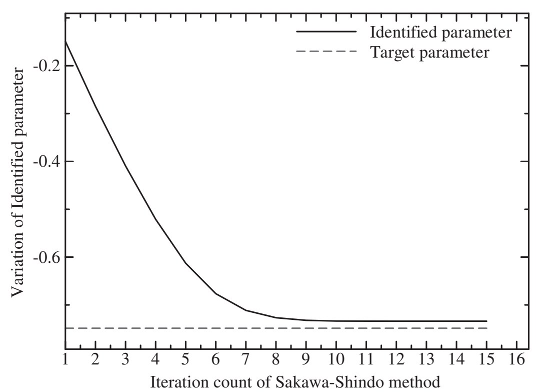

Figure 3. Variation of identified parameter.

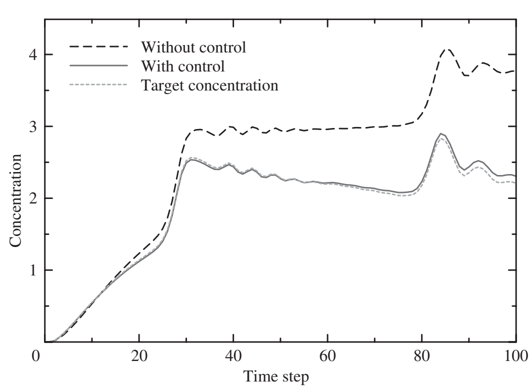

Figure 4. Variation of concentration at objective point.

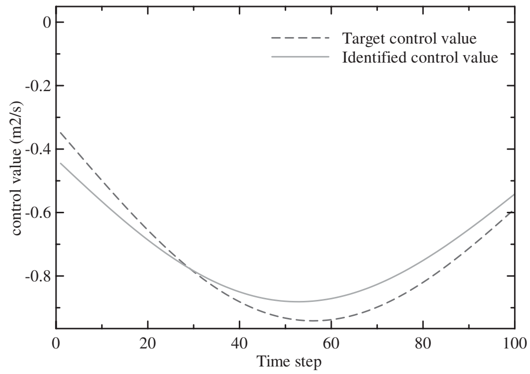

Figure 5. Variation of water conduction value.

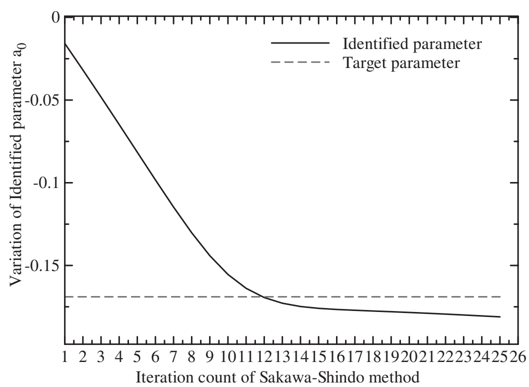

Figure 6. Variation of identified parameter a_0.

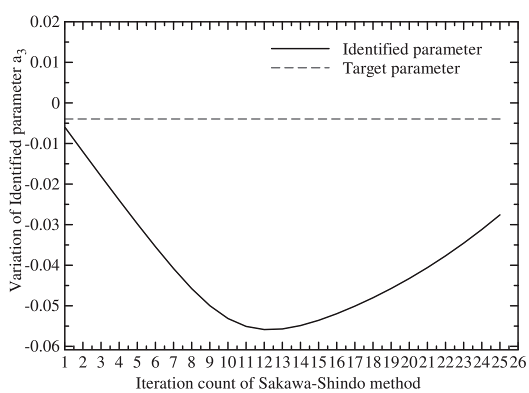

Figure 7. Variation of identified parameter a_3.

Table I. Target and identified parameter.

Parameter	Target	Identified
a_0	0.16894645	0.18105666
a_1	0.14585334	0.15909170
a_2	0.08423875	0.10113634
a_3	0.00395492	0.02758947
a_4	0.07098290	0.03736682
b_1	0.08348577	0.07504478
b_2	0.14880459	0.13181349
b_3	0.17464931	0.14987622
b_4	0.16019777	0.12978499
b_5	0.11815313	0.08587758

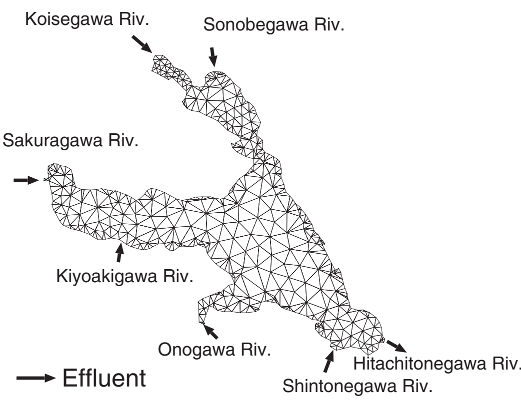

Figure 8. Finite element mesh and river boundary.

which is acceptable. However, the maximum error of the identified parameter is 597.59% which is outside of the acceptable range. The target and identified parameters are shown in Table I.

$$\bar{u}_i(t) = a_0 + \sum_{n=1}^{4} \left\{ a_n \cos\left(\frac{2\pi n}{8T}\, t\right) + b_n \sin\left(\frac{2\pi n}{8T}\, t\right) \right\} + b_5 \sin\left(\frac{2\pi 5}{8T}\, t\right) \tag{17}$$

6.2. Lake Kasumigaura water conduction project

From the numerical results of the rectangular model it is clear that the present method is effective for the optimal control of material concentration. Therefore, the method is applied to an actual civil engineering problem to test the effect on large scale problem (long term calculation), which is the lake Kasumigaura water conduction problem. The Lake Kasumigaura is one of the most polluted lakes in Japan. The water pollution damages both the drinking water and fisheries. To solve the water pollution problem, many projects were started to clean up the water quality in the Lake Kasumigaura. Figure 8 shows the finite element mesh of the Lake Kasumigaura and the river boundary. Plate 2 shows the water depth, and Figure 9

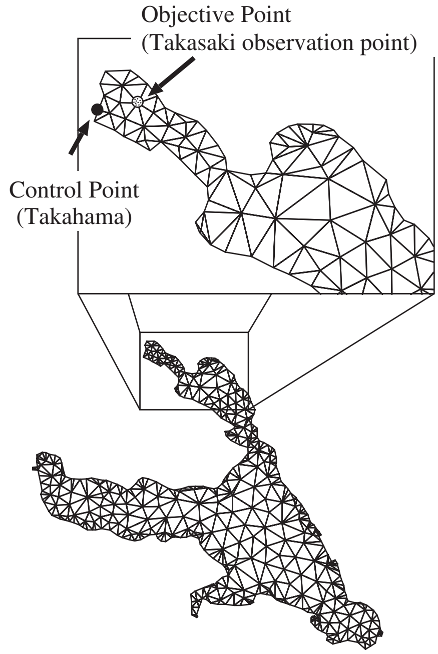

Figure 9. Objective and water conduction point.

Table II. Condition of calculation.

Load value of COD	22.4 (t/day)
Objective concentration of COD	6.1 (mg/l)
Total node	500
Total element	711
Degree of freedom of variable	1211
Time increments Δt	30 (s)
Total time	5 (day)
Time step	14 400 (step)
Diffusion coefficient	0.5 (m^2/s)
Maximum water conduction value	40 (m^3/s)
Minimum water conduction value	0 (m^3/s)

shows the objective point, i.e. Takasaki observation point of water pollution and the control point i.e. water conduction point: Takahama.

The calculation condition is shown in Table II.

Considering the actual maximum quantity of conducted water is 35 (m^3/s), the maximum quantity of water is set to 40 (m^3/s) in the present method. Based on a 1992 water presentation plan for the Lake Kasumigaura, objective COD is assigned as 6.1 (mg/l). In this case, it is assumed that the optimal control quantity is expressed by Fourier series composed of

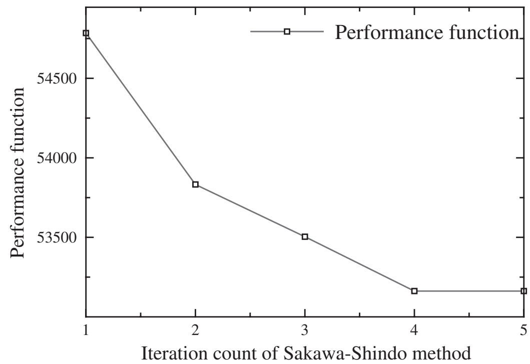

Figure 10. Performance function.

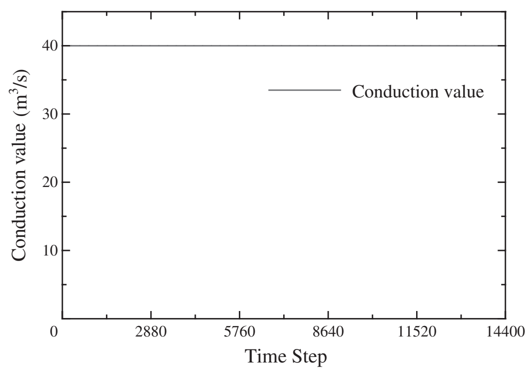

Figure 11. Identified conduction value.

16 terms.

$$Q(t) = a_0 + \sum_{n=1}^{7} \left\{ a_n \cos\left(\frac{2\pi nt}{T}\right) + b_n \sin\left(\frac{2\pi nt}{T}\right) \right\} + b_8 \sin\left(\frac{2\pi 8t}{T}\right) \tag{18}$$

The initial COD condition of the Lake Kasumigaura is calculated by eigenvalue analysis [5]. Plate 3 shows the calculated initial condition and Figure 10 shows the variation of the performance function. The performance function decreases until the fourth step of the Sakawa–Shindo method. Then, at the fifth step, the performance function converges. As a result of this convergence, the identified water conduction value reaches to the maximum. Therefore, the performance function does not decrease any more and the COD concentration does not achieve the objective concentration 6.1 (mg/l). The variation of quantity from the control point and COD concentration at the objective point are shown in Figures 11 and 12, respectively. Plates 4 and 5 show the COD concentration over the whole domain, and Plates 6 and 7 around control point 5 days later. From these results, one cannot hope that water quality will be cleared up all at once with only a water conduction project. However, Plate 7 shows the COD concentration around the control point seems cleared up. This shows that the water

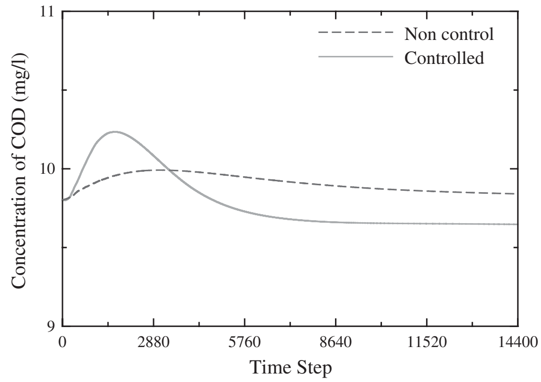

Figure 12. COD concentration at objective point.

Table III. Computational storage requirement.

Procedure	Memory storage
Optimal control theory (2.5 days)	2,008 (M byte)
This research (term number = 16)	1 (M byte)
This research (term number = 160)	11 (M byte)
This research (term number = 1600)	109 (M byte)

conduction, combined with other water purification projects such as dredging and sewage maintenance, is an important part to clear up the water.

The methods to reduce the computational storage requirement is examined, which is an important objective of this research. Table III shows a comparison of computational storage requirements based on normal optimal control theory and the method presented in this paper. A part of optimal control theory shows the computational storage requirement to calculate 2.5 day control case. The computational storage requirement is dependent on the total time cycle in case of optimal control theory. Therefore, in case that computational storage is required to calculate a 5 day control, it is impossible even to use the compilers IBM XL Fortran for AIX and g77 ver.2.95.2-13. It is, however, possible to meet the required computational storage in order to calculate for the duration of 2.5 days. From compiling the result, 2G bytes of storage is needed to calculate the 2.5 day control problem. It is very difficult to calculate such problem without parallel computing, but in the theory of this research, the computational storage requirement is not dependent on the total time cycle. It is also dependent on the term number of Fourier series. Therefore, it is possible to calculate a problem which spans a long period of time. We have confirmed that it is possible to calculate a long term step control problem using Fourier series and parameter identification technique.

7. CONCLUSION

In this paper, a new optimal control theory is presented. For the consideration of this new control theory, it was assumed that an arbitrary function can be expressed by Fourier series.

Ordinarily, the inverse problem requires a plenty of computational storage. However, in the present method, Fourier series and parameter identification technique are employed to calculate the large scale inverse problem. The present method requires significant computational time dependent on the term number of Fourier series. Therefore, a very important future work will be reducing the computational time.

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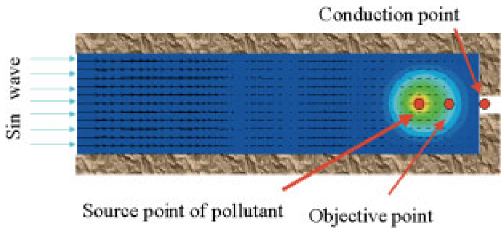

Plate 1. Image of numerical example.

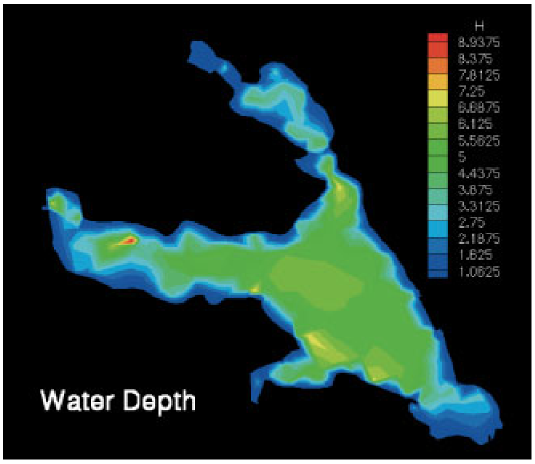

Plate 2. Water depth.

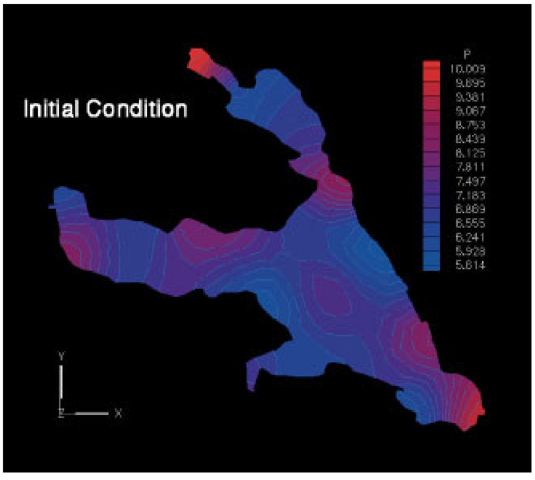

Plate 3. Initial condition of COD (mg/l).

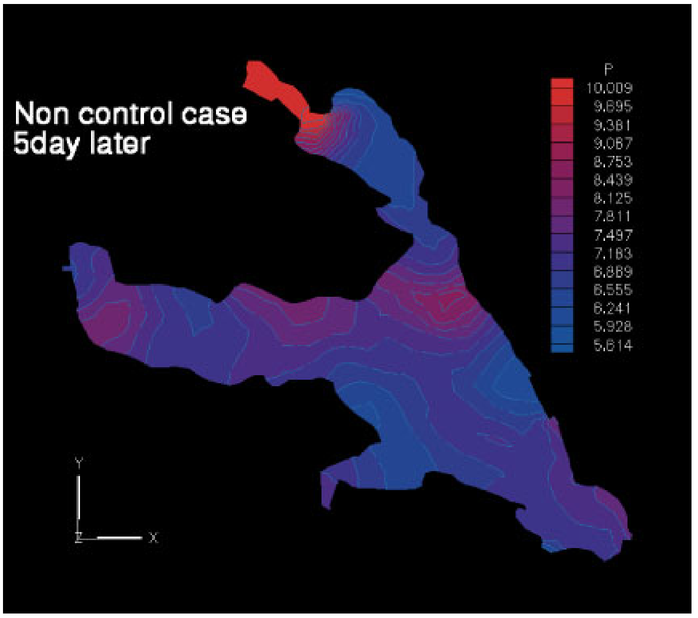

Plate 4. COD concentration (non-control case—Kasumigaura 5 days later).

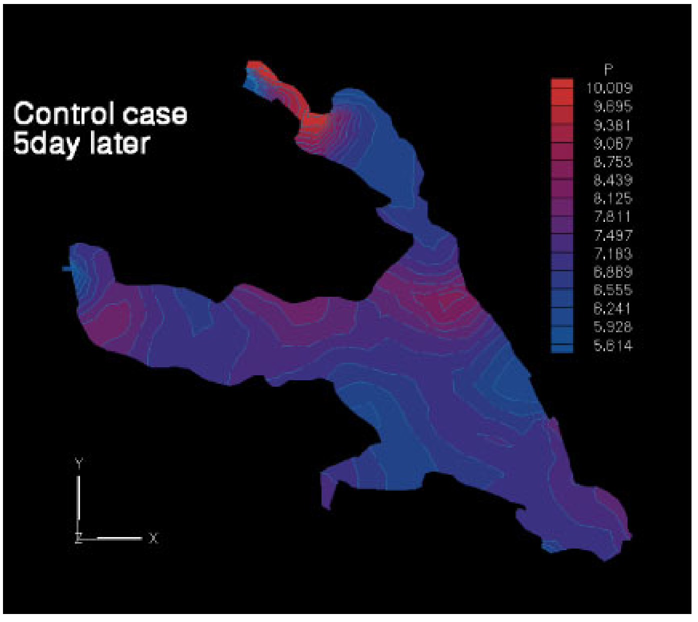

Plate 5. COD concentration (controlled case—Kasumigaura 5 days later).

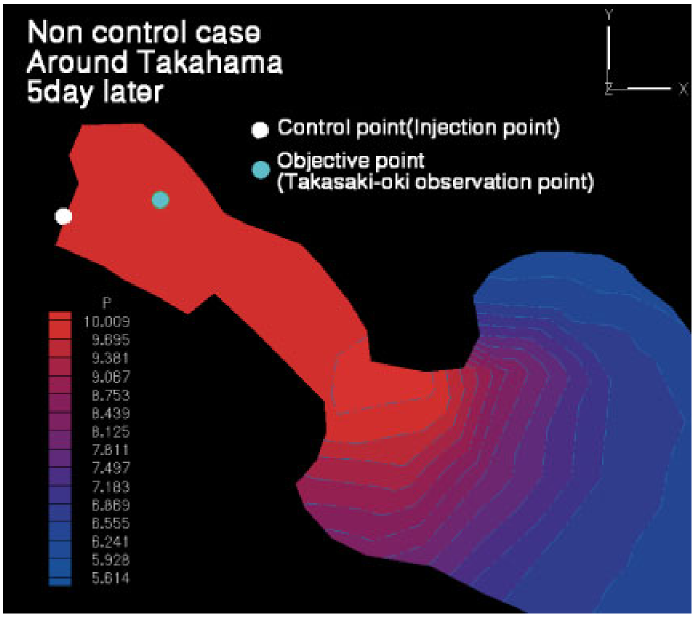

Plate 6. COD concentration (non-control case—around Takahama 5 days later).

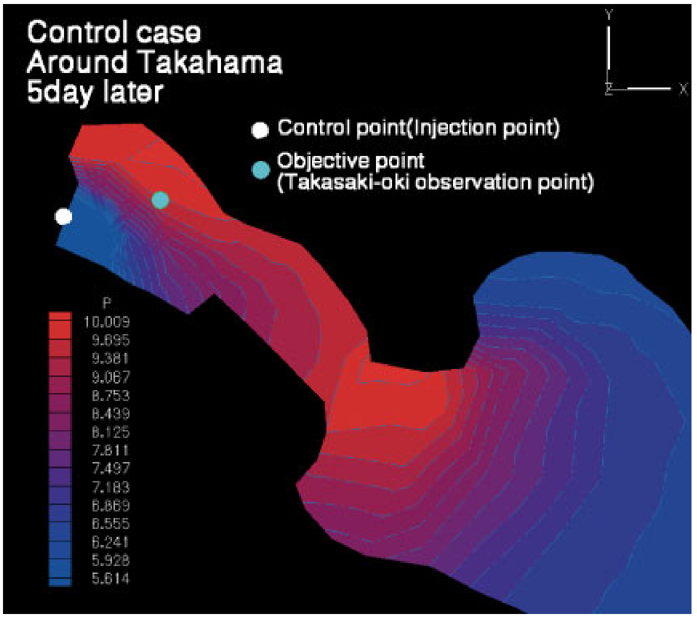

Plate 7. COD concentration (controlled case—around Takahama 5 days later).

